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On the stable containment of two sets

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Abstract This paper studies the stability of the set containment problem. Given two non-empty sets in the Euclidean space which are the solution sets of two systems of (possibly infinite) inequalities, the Farkas type results allow to decide whether one of the two sets is contained or not in the other one (which constitutes the so-called containment problem). In those situations where the data (i.e., the constraints) can be affected by some kind of perturbations, the problem consists of determining whether the relative position of the two sets is preserved by sufficiently small perturbations or not. This paper deals with this stability problem as a particular case of the maintaining of the relative position of the images of two set-valued mappings; first for general set-valued mappings and second for solution sets mappings of convex and linear systems. Thus the results in this paper could be useful in the postoptimal analysis of optimization problems with inclusion constraints.

Keywords Stability theory · Containment problem · Set-valued mappings · Semi-infinite systems

AMS Classification 49K40 · 90C25 · 90C34

1 Introduction

The set containment problem consists of deciding, given two systems in $\mathbb{R}^n$ y_0 and z_0, whether the corresponding solution sets, F_0 and G_0, satisfy $F_0 \subset G_0$ or not. A practical application of this set containment problem is the design centering problem which considers a container

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set $C \subset \mathbb{R}^n$ and a parametrized body $B(\lambda) \subset \mathbb{R}^n$, with parameter $\lambda \in \Lambda$. The idea is to maximize some functional (e.g., the volume of $B(\lambda)$) on the set $\{\lambda \in \Lambda : B(\lambda) \subset C\}$ (see [13]); the problem of cutting a diamond with accepted form and maximal volume from a raw diamond is an example. The multi-body design centering problem considers $B(\lambda)$ to be a finite union of non-overlapping bodies (with finitely many connected components). It is also possible to consider the question about if, given $\bar{\lambda} \in \Lambda$ such that $B(\bar{\lambda}) \subset C$, then $B(\lambda) \subset C$ for λ close enough to $\bar{\lambda}$. When the sets C and $B(\lambda)$ are described by inequality constraints

$$C = \left\{x \in \mathbb{R}^n : c_s(x) \leq 0, s \in S\right\}$$

and

$$B(\lambda) = \left\{x \in \mathbb{R}^n : b_j(x) \leq 0, j \in J; \quad \omega_\lambda(x) \leq 0\right\}$$

for some sets of indexes S and J, while $\lambda \in \Lambda$, then one can pose the problem of studying the stability of the inclusion $B(\lambda) \subset C$ when some of the data $(c_s, b_j, \omega_\lambda)$ are subject to small perturbations.

Another application is the knowledge-based data classification problem: F_0 represents the knowledge set (which is fixed) whereas G_0 belongs to a predetermined family of sets from which the decision maker selects one of the best for a certain optimality criterion under the condition that $F_0 \subset G_0$. This problem was posed in [4] (where F_0 is a given convex polyhedron and G_0 is required to be a half-space) and extended by Mangasarian, Jeyakumar et al. to more general situations: in [10] and [11], where F_0 and G_0 are solution sets of ordinary systems of differentiable constraints; in [8], which deals with ordinary convex systems and linear semi-infinite systems (in brief, LSISs) and, finally, in [5], where also convex semi-infinite systems (CSISs) are considered. The last paper considers the stable containment problem for LSISs, i.e., deciding whether the inclusion of solution sets of such kind of systems is preserved by sufficiently small perturbation of the coefficients.

This paper extends the stable containment problem to CSISs, to general semi-infinite systems and even to arbitrary set-valued mappings.

More in detail, we consider given two set-valued mappings $\mathcal{F} : Y \rightrightarrows \mathbb{R}^n$ and $\mathcal{G} : Z \rightrightarrows \mathbb{R}^n$, where (Y, ρ_Y) and (Z, ρ_Z) are pseudometric spaces and a couple $(y_0, z_0) \in Y \times Z$.

We say that the *containment* $\mathcal{F}(y_0) \subset \mathcal{G}(z_0)$ is *stable* (in brief, $\mathcal{F} \subset \mathcal{G}$ stably) at (y_0, z_0) if there exists $\varepsilon > 0$ such that $\mathcal{F}(y) \subset \mathcal{G}(z)$ for all $(y, z) \in Y \times Z$ with $\rho_Y(y, y_0) < \varepsilon$ and $\rho_Z(z, z_0) < \varepsilon$.

In the case of systems, we consider given $y_0 = \left\{f_t^0(x) \leq 0, t \in T\right\}$ and $z_0 = \left\{g_s^0(x) \leq 0, s \in S\right\}$, where the index sets T and S are arbitrary and $f_t^0, g_s^0 : \mathbb{R}^n \to \mathbb{R}$ are bounded functions on bounded sets, for all $t \in T$ and for all $s \in S$. We denote by Y the class of all systems of the form $y = \{f_t(x) \leq 0, t \in T\}$ (i.e., those systems which have the same space of variables, $\mathbb{R}^n$, and the same index set, T) and, in a similar way, the space of parameters associated with z_0, say Z. We will identify each system $\{f_t(x) \leq 0, t \in T\}$ with its corresponding set of data (functions) $\{f_t\}_{t \in T}$. In Sect. 3 below we define pseudometrics on the parameter spaces Y and Z via these sets of data; these pseudometrics provide the uniform convergence on T and S, respectively. So, in this paper systems of inequalities are view as points in pseudometric spaces. In this context $\mathcal{F}$ and $\mathcal{G}$ will be the feasible set mappings, i.e., $\mathcal{F}(y)$ and $\mathcal{G}(z)$ are the solution sets of $y = \{f_t(x) \leq 0, t \in T\}$ and $z = \{g_t(x) \leq 0, s \in S\}$, respectively.

We recall the stability concepts and some basic results for set-valued mappings that we shall consider in this paper. Let $\mathcal{F} : Y \rightrightarrows \mathbb{R}^n$ be a set-valued mapping. Its domain is $\operatorname{dom} \mathcal{F} := \{y \in Y : \mathcal{F}(y) \neq \emptyset\}$. The following semicontinuity concepts are due to Bouligand and Kuratowski (see [1, Sect. 1.4]).

We say that $\mathcal{F}$ is *lower semicontinuous* at $y_0 \in Y$ (lsc, in brief) if, for each open set $W \subset \mathbb{R}^n$ such that $W \cap \mathcal{F}(y_0) \neq \emptyset$, there exists an open set $V \subset Y$, containing y_0, such that $W \cap \mathcal{F}(y) \neq \emptyset$ for each $y \in V$. Obviously, $\mathcal{F}$ is lsc at $y_0 \notin \operatorname{dom} \mathcal{F}$ and $y_0 \in \operatorname{int} \operatorname{dom} \mathcal{F}$ if $\mathcal{F}$ is lsc at $y_0 \in \operatorname{dom} \mathcal{F}$.

$\mathcal{F}$ is *upper semicontinuous* at $y_0 \in Y$ (usc, in brief) if, for each open set $W \subset \mathbb{R}^n$ such that $\mathcal{F}(y_0) \subset W$, there exists an open set $V \subset Y$, containing y_0, such that $\mathcal{F}(y) \subset W$ for each $y \in V$. Clearly, if $\mathcal{F}$ is usc at $y_0 \notin \operatorname{dom} \mathcal{F}$, then $y_0 \in \operatorname{int} (Y \backslash \operatorname{dom} \mathcal{F})$.

If $\mathcal{F}$ is simultaneously lsc and usc at y_0 we say that $\mathcal{F}$ is *continuous* at this point.

$\mathcal{F}$ is *closed* at $y_0 \in \operatorname{dom} \mathcal{F}$ if for all sequences $\{y_r\} \subset Y$ and $\{x_r\} \subset \mathbb{R}^n$ satisfying $x_r \in \mathcal{F}(y_r)$ for all $r \in \mathbb{N}$, $y_r \to y_0$, and $x_r \to x_0$, one has $x_0 \in \mathcal{F}(y_0)$. If $\mathcal{F}$ is usc at $y_0 \in \operatorname{dom} \mathcal{F}$ and $\mathcal{F}(y_0)$ is closed, then $\mathcal{F}$ is closed at y_0. Conversely, if $\mathcal{F}$ is closed and *locally bounded* at $y_0 \in \operatorname{dom} \mathcal{F}$ (i.e., if there are a neighborhood of y_0, say V, and a bounded set $A \subset \mathbb{R}^n$ containing $\mathcal{F}(y)$ for every $y \in V$), then $\mathcal{F}$ is usc at y_0.

$\mathcal{F}$ is lsc (usc, closed, locally bounded) if it is lsc (usc, closed, locally bounded) at y for all $y \in Y$.

The *boundary mapping* of $\mathcal{F}$ is $\operatorname{bd} \mathcal{F} : Y \rightrightarrows \mathbb{R}^n$ such that $(\operatorname{bd} \mathcal{F})(y) = \operatorname{bd} \mathcal{F}(y)$ for all $y \in Y$. (Here $\operatorname{bd} \mathcal{F}(y)$ stands for the boundary of the set $\mathcal{F}(y)$). In [7] it has been shown that $\operatorname{bd} \mathcal{F}$ is usc at y_0 if $\mathcal{F}$ is usc at y_0 and $\mathcal{F}(y)$ is the convex hull of $\operatorname{bd} \mathcal{F}(y)$ for y close to y_0.

Other notions of lower and upper semicontinuity as lsc and usc in the sense of Hausdorff (see, e.g., [2]) or inner and outer semicontinuity (see, e.g., [12], where it is shown that the last two concepts are equivalent to lsc and closedness when $\mathcal{F}$ is closed-valued) will not be considered in particular in this paper.

As an illustrative example, consider two particular instances of the design centering problem:

(a) Determining the greatest closed ball contained in C. Here $Y = \mathbb{R}^3 \times \mathbb{R}_{++}$ and, given $y = (y_1, y_2) \in Y$ y_1 represents the center of the ball and y_2 its radius.
(b) Determining the greatest object of a prescribed form (fixed by means of some compact pattern set) contained in C. Now the decision variables are the position of the gravity center (translation), the orientation of the selected axis (rotation), and the scale factor (dilation), so that $Y \subset \mathbb{R}^7$.

In both cases $\mathcal{G}$ is the constant mapping $\mathcal{G}(z) = C$ (for any pseudometric space Z) and both set-valued mappings, $\mathcal{F}$ and $\mathcal{G}$, are obviously lsc, usc, and closed.

Finally, some additional notation: the Euclidean and the Tchebyshev norms in $\mathbb{R}^n$ are denoted by $\|\cdot\|$ and $\|\cdot\|_\infty$, respectively. The Euclidean open ball centered at x and radius $r > 0$ is represented by $B(x; r)$. 0_n is the null vector in $\mathbb{R}^n$. For any subsets $A, B \subset \mathbb{R}^n$ $d(A, B)$ denotes $\inf \{\|x - y\| : x \in A,\ y \in B\}$, with $\inf \emptyset = +\infty$. If A is a subset of $\mathbb{R}^n$ $\operatorname{int} A$ and $\operatorname{cl} A$ represent the interior and the closure of A, respectively.

The paper is organized as follows. Section 2 deals with arbitrary set-valued mappings, including conditions for $\mathcal{F} \subset \mathcal{G}$ stably at (y_0, z_0), which involve semicontinuity properties of $\mathcal{F}$ at y_0 and $\mathcal{G}$ at z_0, together with geometric conditions like $\mathcal{F}(y_0) \subset \operatorname{int} \mathcal{G}(z_0)$ or $d[\mathcal{F}(y_0), \operatorname{bd} \mathcal{G}(z_0)] > 0$. Taking into account the generality of the set-valued mappings considered in this section, it is not surprising that we need to impose several hypotheses in order to get some sufficient or necessary conditions for the stability of the set containment. Section 3 exploits the results developed in Sect. 2 for obtaining analogous properties for semi-infinite systems. Since this section considers specially structured set-valued mappings by focussing the attention on the solution sets of CSISs and LSISs, the hypotheses discussed in the previous section are now quite easy to state.

2 Set-valued mappings

Throughout this section we assume that $\mathcal{F} : Y \rightrightarrows \mathbb{R}^n$ and $\mathcal{G} : Z \rightrightarrows \mathbb{R}^n$ are two given set-valued mappings, with (Y, ρ_Y) and (Z, ρ_Z) pseudometric spaces, and $(y_0, z_0) \in Y \times Z$.

We show in this section that, under suitable assumptions, $d\left[\mathcal{F}(y_0), \text{ bd}\,\mathcal{G}(z_0)\right] > 0$ implies that $\mathcal{F} \subset \mathcal{G}$ stably at (y_0, z_0) whereas the latter property entails $\mathcal{F}(y_0) \subset \text{ int}\,\mathcal{G}(z_0)$.

Lemma 2.1 *Let* $(y_0, z_0) \in Y \times Z$ *such that* $\mathcal{F}(y_0) \subset \mathcal{G}(z_0)$ *and there exists an open connected set* U *such that* $\mathcal{F}(y_0) \subset U \subset \mathcal{G}(z_0)$. *Then*

(i) $d\left[U, \text{ bd}\,\mathcal{G}(z_0)\right] > 0$,
(ii) $\mathcal{F}$ *is usc at* y_0,
(iii) $\mathcal{G}$ *is lsc at* z_0, *and*
(iv) bd $\mathcal{G}$ *is usc at* z_0,

imply that $\mathcal{F} \subset \mathcal{G}$ *stably at* (y_0, z_0).

Proof If $\mathcal{F}(y_0) = \emptyset$, by (ii) we have $\mathcal{F}(y) = \emptyset$ in a certain neighborhood of y_0 . Thus we assume that $\mathcal{F}(y_0) \neq \emptyset$. If $\mathcal{G}(z_0) = \mathbb{R}^n$ and so bd $\mathcal{G}(z_0) = \emptyset$, by (iii) and (iv) $\mathcal{G}(z) = \mathbb{R}^n$ for z close enough to z_0. In the following discussion we confine ourselves to the non-trivial cases in which $\mathcal{F}(y_0) \neq \emptyset$ and $\mathcal{G}(z_0) \neq \mathbb{R}^n$.

Let $V := \mathbb{R}^n \backslash \text{cl}U$. Obviously, $U \cap V = \emptyset$ and bd $\mathcal{G}(z_0) \subset V$ by (i).

Since $\mathcal{F}(y_0) \subset U$, by (ii), there exists $\varepsilon_1 > 0$ such that

$$\mathcal{F}(y) \subset U \quad \text{if } \rho_Y(y, y_0) < \varepsilon_1. \tag{2.1}$$

As $\emptyset \neq \mathcal{F}(y_0) \subset \mathcal{G}(z_0) \cap U$ and $\mathcal{G}$ is lsc at z_0, there exists $\varepsilon_2 > 0$ such that

$$\mathcal{G}(z) \cap U \neq \emptyset \quad \text{if } \rho_Z(z, z_0) < \varepsilon_2. \tag{2.2}$$

Because bd $\mathcal{G}(z_0) \subset V$ and *(iv)* holds, there exists $\varepsilon_3 > 0$ such that

$$\text{bd}\,\mathcal{G}(z) \subset V \quad \text{if } \rho_Z(z, z_0) < \varepsilon_3. \tag{2.3}$$

Finally we prove that $\mathcal{F}(y) \subset \mathcal{G}(z)$ if $\rho_Y(y, y_0) < \varepsilon$ and $\rho_Z(z, z_0) < \varepsilon$, for $\varepsilon := \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$. By (2.1), it is enough to show that $U \subset \mathcal{G}(z)$ for all $z \in Z$ such that $\rho_Z(z, z_0) < \varepsilon$.

Let us assume the contrary, i.e., $U \nsubseteq \mathcal{G}(z)$. Take $x_1 \in U \backslash \mathcal{G}(z)$. By (2.2), we can take also $x_2 \in \mathcal{G}(z) \cap U$. Since U is an open connected set, there exists a continuous mapping $\alpha : [0, 1] \rightarrow U$ such that $\alpha(0) = x_1$ and $\alpha(1) = x_2$. Let

$$\bar{t} := \sup\{t \in [0, 1] : \alpha(t) \in \mathcal{G}(z)\}.$$

Then $x_3 := \alpha\left(\bar{t}\right) \in [\text{ bd}\,\mathcal{G}(z)] \backslash V$ because $U \cap V = \emptyset$, in contradiction with (2.3). □

Theorem 2.2 *Let* $(y_0, z_0) \in Y \times Z$ *such that* $\mathcal{F}(y_0) \subset \mathcal{G}(z_0)$ *and*

(i) $d\left[\mathcal{F}(y_0), \text{ bd}\,\mathcal{G}(z_0)\right] > 0$,
(ii) $\mathcal{F}$ *is usc at* y_0,
(iii) $\mathcal{G}$ *is lsc at* z_0, *and*
(iv) bd $\mathcal{G}$ *is usc at* z_0.

Then, any of the following three conditions guarantees that $\mathcal{F} \subset \mathcal{G}$ stably at (y_0, z_0):

(a) $\mathcal{F}(y_0)$ *is bounded.*
(b) $\mathcal{F}(y_0)$ *has finitely many connected components.*
(c) $\operatorname{int} \mathcal{G}(z_0)$ *is a convex set.*

Proof We assume $\mathcal{F}(y_0) \neq \emptyset$. Take $\mu := \frac{1}{2} d\left[\mathcal{F}(y_0), \ \operatorname{bd} \mathcal{G}(z_0)\right]$.

First suppose that $\mathcal{F}(y_0)$ is bounded. Consider the open bounded sets

$$U_1 := \mathcal{F}(y_0) + B(0_n; \mu/2) \subset U_2 := \mathcal{F}(y_0) + B(0_n; \mu) \subset \operatorname{int} \mathcal{G}(z_0).$$

Since $\mathcal{F}(y_0) \subset U_1$ and $\operatorname{bd} \mathcal{G}(z_0) \subset V := \mathbb{R}^n \setminus \operatorname{cl} U_2$, there exists $\varepsilon > 0$ such that

$$\mathcal{F}(y) \subset U_1 \quad \text{if } \rho_Y(y, y_0) < \varepsilon$$

and

$$\operatorname{bd} \mathcal{G}(z) \subset V \quad \text{if } \rho_Z(z, z_0) < \varepsilon.$$

If $U_1 \subset \mathcal{G}(z)$ for all z close enough to z_0, then we are done. Otherwise, there exists a sequence $\{z_r\}$ such that $z_r \to z_0$ and $U_1 \nsubseteq \mathcal{G}(z_r)$ for all $r = 1, 2, \ldots$. Take $x_r \in U_1 \backslash \mathcal{G}(z_r)$ and assume without loss of generality that $x_r \to \bar{x}$. Then $\bar{x} \in \operatorname{cl} U_1 \subset U_2$ and we may consider some open ball B centered at $\bar{x}$ contained in U_2. Since $B \cap \mathcal{G}(z_0) \neq \emptyset$, condition (iii) implies that $B \cap \mathcal{G}(z_r) \neq \emptyset$ for r large enough. Let r_0 be such that $x_r \in B$, $B \cap \mathcal{G}(z_r) \neq \emptyset$ and $\operatorname{bd} \mathcal{G}(z_r) \subset V$ for all $r \geq r_0$. The fact that B is connected yields a contradiction as in the proof of the previous lemma.

Second assume that $\mathcal{F}(y_0)$ is connected. The set $U := \mathcal{F}(y_0) + B(0_n; \mu)$ is open and connected. Since $d\left[U, \ \operatorname{bd} \mathcal{G}(z_0)\right] = \mu > 0$, Lemma 2.1 applies.

In the case that $\mathcal{F}(y_0)$ has finitely many connected components, say F_i, $i = 1, \ldots, k$, we only need to replace ε_2 in the proof of the previous lemma by $\varepsilon_2 = \min_{i=1,\ldots,k} \eta_i$, where each $\eta_i > 0$ is such that $\mathcal{G}(z) \cap (F_i + B(0_n; \mu)) \neq \emptyset$ if $\rho_Z(z, z_0) < \eta_i$ and consider an appropriate connected component.

Finally we assume that $\operatorname{int} \mathcal{G}(z_0)$ is a convex set. The set

$$U := \{x \in \operatorname{int} \mathcal{G}(z_0) : d\left[x, \ \operatorname{bd} \mathcal{G}(z_0)\right] > \mu\}$$

is open and satisfies $\mathcal{F}(y_0) \subset U$. Now we prove that U is convex. Let $x_i \in U$, $i = 1, 2$. Since $\operatorname{int} \mathcal{G}(z_0)$ is convex and $B(x_i; \mu) \subset \operatorname{int} \mathcal{G}(z_0)$ $i = 1, 2$, we have

$$\bigcup_{\lambda \in [0,1]} \{(1 - \lambda) B(x_1; \mu) + \lambda B(x_2; \mu)\} \subset \operatorname{int} \mathcal{G}(z_0),$$

so that $d\left[(1 - \lambda) x_1 + \lambda x_2, \ \operatorname{bd} \mathcal{G}(z_0)\right] > \mu$ for all $\lambda \in [0, 1]$. Hence $[x_1, x_2] \subset U$ and Lemma 2.1 applies again. □

Corollary 2.3 *Let $\mathcal{G}$ be closed-convex-valued and let $(y_0, z_0) \in Y \times Z$. If $\mathcal{F}(y_0)$ is a closed convex subset of $\operatorname{int} \mathcal{G}(z_0)$, $\mathcal{G}(z_0)$ is bounded, $\mathcal{F}$ is usc at y_0 and $\mathcal{G}$ is continuous at z_0, then $\mathcal{F} \subset \mathcal{G}$ stably at (y_0, z_0).*

Proof Under the assumptions $\mathcal{G}$ is locally bounded at z_0, so that the convex hull of $\operatorname{bd} \mathcal{G}$ coincides with $\mathcal{G}$ in a certain neighborhood of z_0. Thus $\operatorname{bd} \mathcal{G}$ is continuous. On the other hand, since $\mathcal{F}(y_0)$ and $\operatorname{bd} \mathcal{G}(z_0)$ are disjoint compact sets, $d\left[\mathcal{F}(y_0), \ \operatorname{bd} \mathcal{G}(z_0)\right] > 0$. □

Example 2.4 The next four cases show that none of the four conditions (i)–(iv) in Theorem 2.2 is superfluous. In all of them $\mathcal{F}$ and $\mathcal{G}$ are solution sets mappings corresponding to linear systems in $\mathbb{R}^2$ (so that they are closed-convex-valued). The sets Y and Z are endowed with the usual topology in $\mathbb{R}$.
(a) Let $Y = Z = \mathbb{R}_{++}$,

$$\mathcal{F}(y) = \left\{x \in \mathbb{R}^2 : -y \leq x_i \leq y, i = 1, 2\right\},$$

$\mathcal{G} = \mathcal{F}$ and $y_0 = z_0 = 1$. $\mathcal{F}$ (and so $\mathcal{G}$) is continuous, so that (ii)–(iv) hold. Nevertheless $\mathcal{F}(y) \nsubseteq \mathcal{G}(z)$ if $z < 1 < y$.
(b) Let $Y = Z = \mathbb{R}_+$,

$$\mathcal{F}(y) = \left\{x \in \mathbb{R}^2 : x_1 + yx_2 \geq 0, -x_1 + yx_2 \geq 0\right\},$$

$$\mathcal{G}(z) = \left\{x \in \mathbb{R}^2 : -1 \leq x_1 \leq 1\right\}$$

(constant) and $y_0 = z_0 = 0$. Now (i), (iii) and (iv) hold whereas $\mathcal{F}(y) \nsubseteq \mathcal{G}(z_0)$ if $y > 0$.
(c) Let $Y = Z = \mathbb{R}$,

$$\mathcal{F}(y) = \left\{x \in \mathbb{R}^2 : -1 \leq x_i \leq 1, i = 1, 2\right\}$$

(constant),

$$\mathcal{G}(z) = \left\{x \in \mathbb{R}^2 : -2 \leq x_i \leq 2, i = 1, 2; 0_2' x \leq z\right\}$$

and $y_0 = z_0 = 0$. Obviously, (i), (ii) and (iv) hold but $\mathcal{F}(0) = [-1, 1]^2 \nsubseteq \mathcal{G}(z) = \emptyset$ if $z < 0$.
(d) Let $Y = Z = [0, 1]$ $\mathcal{F}(y) = \left\{x \in \mathbb{R}^2 : x_i \geq 1, i = 1, 2\right\}$ (constant),

$$\mathcal{G}(z) = \left\{x \in \mathbb{R}^2 : -x_1 + zx_2 \leq 0, zx_1 - x_2 \leq 0\right\}$$

and $y_0 = z_0 = 0$. It is easy to see that (i)–(iii) hold but $\mathcal{F}(y_0) \nsubseteq \mathcal{G}(z)$ if $0 < z < 1$.

It is an open problem to know whether the alternative conditions (a), (b), (c) in Theorem 2.2 are superfluous or not.

In [6] it is shown the existence of consistent LSISs such that the solution set is constant under arbitrary but sufficiently small perturbation of the coefficients (i.e., $\mathcal{F}$ is constant in a certain neighborhood of y_0), so we can have $\mathcal{F} \subset \mathcal{G}$ stably at (y_0, z_0) and nevertheless $\mathcal{F}(y_0) \nsubseteq \operatorname{int} \mathcal{G}(z_0)$ (take for instance $\mathcal{G} = \mathcal{F}$). The next result provides conditions guaranteeing that the inclusion $\mathcal{F}(y_0) \subset \operatorname{int} \mathcal{G}(z_0)$ is necessary for the stability of $\mathcal{F} \subset \mathcal{G}$ at (y_0, z_0). The intuitive meaning of such conditions are that $\mathcal{F}$ expands in the proximity of y_0 and $\mathcal{G}$ shrinks inwards close to z_0, respectively.

Condition A: $\mathcal{F}$ satisfies that for all closed half-space S such that $\mathcal{F}(y_0) \subset S$ and bd S supports $\mathcal{F}(y_0)$, there exists a sequence $\{y_r\} \subset Y$ such that $y_r \to y_0$ and

$$\mathcal{F}(y_r) \setminus S \neq \emptyset \quad \text{for all } r \in \mathbb{N}.$$

Condition B: $\mathcal{G}$ satisfies that for all supporting hyperplanes to $\mathcal{G}(z_0)$, say H, there exists $\{z_r\} \subset Z$ such that $z_r \to z_0$ and

$$\mathcal{G}(z_r) \cap H = \emptyset \quad \text{for all } r \in \mathbb{N}.$$

Remarks 3.1 and 3.2 in Sect. 3 show sufficient conditions (related to an equilipschitzian property) for Conditions **A** and **B** in the case of inequality systems. The following examples illustrate them for parametric systems:

Example 2.5 Let $n = 2,\ Y = [1, +\infty[\ \ Z =]0, 1]$ with the usual topology, $y_0 = z_0 = 1$,

$$\mathcal{F}(y) = \left\{x \in \mathbb{R}^2 : (\cos t)\, x_1 + (\sin t)\, x_2 \le y, 0 \le t \le 2\pi\right\}$$

and

$$\mathcal{G}(z) = \left\{x \in \mathbb{R}^2 : (\cos t)\, x_1 + (\sin t)\, x_2 \le z, 0 \le t \le 2\pi\right\}.$$

Then $\mathcal{F}$ and $\mathcal{G}$ satisfy conditions **A** and **B**, respectively.

Example 2.6 Let M be the set of all symmetric positive definite matrices A of order $n \times n$ and consider the set-valued mapping

$$\mathcal{E} : M \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n, \quad \mathcal{E}(A, b) = \left\{x \in \mathbb{R}^n : \|Ax + b\| \le 1\right\}.$$

Here $E = \mathcal{E}(A, b)$ is an ellipsoid. The two geometric problems in $\mathbb{R}^n$ of finding the Löwner-John ellipsoid of a set C which is a minimum volume ellipsoid E such that $C \subset E$, and of finding the center of a convex set C defined as the center of a maximum volume inscribed ellipsoid are related to this mapping $\mathcal{E}$; actually $\mathcal{E}$ plays the role of $\mathcal{G}$ in the first problem and of $\mathcal{F}$ in the second one. Consider $M \times \mathbb{R}^n$ with the Euclidean topology and let $(A_0, b_0) \in M \times \mathbb{R}^n$. If $c'x = d, c \ne 0_n$, is any supporting hyperplane H to $\mathcal{E}(A_0, b_0)$ with $c'x \le d$ for any $x \in \mathcal{E}(A_0, b_0)$, take $x_0 \in H \cap \mathcal{E}(A_0, b_0)$ and put $(A_r, b_r) = \left(A_0, b_0 - \frac{1}{r} A_0 c\right)$, $\left(\widetilde{A}_r, \widetilde{b}_r\right) = \left(A_0, b_0 + \frac{1}{r} A_0 c\right)$, for $r \in \mathbb{N}$, then $(A_r, b_r) \to (A_0, b_0)$, $x_r = x_0 + \frac{1}{r} c \in \mathcal{E}(A_r, b_r)$, $c'x_r > d$; $\left(\widetilde{A}_r, \widetilde{b}_r\right) \to (A_0, b_0)$ and $c'x < d$ for any x in $\mathcal{E}\left(\widetilde{A}_r, \widetilde{b}_r\right)$. Therefore $\mathcal{E}$ satisfies both conditions **A** and **B**.

Theorem 2.7 *Let $\mathcal{F} \subset \mathcal{G}$ stably at (y_0, z_0) and assume that $\mathcal{G}(z_0)$ is convex and either $\mathcal{F}$ satisfies condition* **A** *or $\mathcal{G}$ satisfies condition* **B**. *Then $\mathcal{F}(y_0) \subset \operatorname{int} \mathcal{G}(z_0)$.*

Proof We assume that $\mathcal{F} \subset \mathcal{G}$ stably at (y_0, z_0) but $\mathcal{F}(y_0) \not\subset \operatorname{int} \mathcal{G}(z_0)$. Let $\bar{x} \in \mathcal{F}(y_0) \setminus \operatorname{int} \mathcal{G}(z_0)$. Then $\bar{x} \in \operatorname{bd} \mathcal{G}(z_0)$, and so by the supporting hyperplane theorem there exists $w \in \mathbb{R}^n \setminus \{0_n\}$ such that

$$w'(x - \bar{x}) \le 0 \quad \text{for all } x \in \operatorname{cl}\mathcal{G}(z_0).$$

Since $\bar{x} \in \mathcal{F}(y_0) \subset \mathcal{G}(z_0)$, the hyperplane

$$H := \left\{x \in \mathbb{R}^n : w'(x - \bar{x}) = 0\right\}$$

also supports $\mathcal{F}(y_0)$ at $\bar{x}$. Put

$$S := \left\{x \in \mathbb{R}^n : w'(x - \bar{x}) \le 0\right\}.$$

We will get a contradiction from both conditions **A** and **B**. If **A** holds, we can take $\{y_r\} \subset Y,\ y_r \to y_0,\ \mathcal{F}(y_r) \setminus S \ne \emptyset$ for all $r \in \mathbb{N}$. Since $\mathcal{G}(z_0) \subset S$, we also have $\mathcal{F}(y_r) \setminus \mathcal{G}(z_0) \ne \emptyset$, so that $\mathcal{F}(y_r) \not\subset \mathcal{G}(z_0)$ for all r.

Alternatively, if **B** holds there exists $\{z_r\} \subset Z,\ z_r \to z_0$, such that $\mathcal{G}(z_r) \cap H = \emptyset$ for all r. Since $\bar{x} \in H$, we have $\bar{x} \notin \mathcal{G}(z_r)$, so that $\bar{x} \in \mathcal{F}(y_0) \setminus \mathcal{G}(z_r)$ and we have $\mathcal{F}(y_0) \not\subset \mathcal{G}(z_r)$ for all r.

In both cases we get a contradiction. □

The conclusion $\mathcal{F}(y_0) \subset \operatorname{int} \mathcal{G}(z_0)$ in the previous theorem cannot be replaced by the stronger one $d(\mathcal{F}(y_0), \operatorname{bd} \mathcal{G}(z_0)) > 0$, as the set-valued mappings $\mathcal{F}$ and $\mathcal{G}$ (both of them solution set mappings corresponding to parametric systems) in the following simple examples show. Nonetheless, in the next section we prove this property for semi-infinite systems under mild assumptions.

Example 2.8 Let $Y = Z =]0, 1]$,

$$\mathcal{F}(y) = \left\{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0, x_1 x_2 \geq y\right\},$$

$\mathcal{G}(z) = \left\{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0\right\}$(constant) and $y_0 = z_0 = 1$. Then $\mathcal{F} \subset \mathcal{G}$ stably at (y_0, z_0), condition **A** is satisfied and $\mathcal{F}(y_0) \subset \operatorname{int} \mathcal{G}(z_0)$ but $d(\mathcal{F}(y_0), \operatorname{bd} \mathcal{G}(z_0)) = 0$.

Example 2.9 Let $Y = Z = [0, 1[$,

$$\mathcal{F}(y) = \left\{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0, x_1 x_2 \geq 1\right\}$$

(constant),

$$\mathcal{G}(z) = \left\{x \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0, x_1 x_2 \geq z\right\}$$

and $y_0 = z_0 = 0$. Then $\mathcal{F} \subset \mathcal{G}$ stably at (y_0, z_0), condition **B** holds and $\mathcal{F}(y_0) \subset \operatorname{int} \mathcal{G}(z_0)$, but once again $d(\mathcal{F}(y_0), \operatorname{bd} \mathcal{G}(z_0)) = 0$.

3 Solution sets of systems

Throughout this section we consider two given (nominal) systems $y_0 = \left\{f_t^0(x) \leq 0, t \in T\right\}$ and $z_0 = \left\{g_t^0(x) \leq 0, s \in S\right\}$ with corresponding spaces of parameters, denoted by Y and Z in the general case, by Y_C and Z_C if y_0 and z_0 are convex systems and by Y_L and Z_L if y_0 and z_0 are linear systems. $\mathcal{F}$ and $\mathcal{G}$ are the feasible set mappings, i.e., $\mathcal{F}(y)$ and $\mathcal{G}(z)$ are the solution sets of $y = \{f_t(x) \leq 0, t \in T\}$ and $z = \{g_t(x) \leq 0, s \in S\}$, respectively.

In the case of linear systems $y_0 = \{(a_t^0)'x \leq b_t^0, t \in T\}$ and $z_0 = \{(c_s^0)'x \leq d_s^0, s \in S\}$, where the index sets T and S are arbitrary and $a_t^0, c_s^0 \in \mathbb{R}^n, b_t^0, d_s^0 \in \mathbb{R}$ for all $t \in T$ and for all $s \in S$. We denote by Y_L the class of all the linear systems of the form $y = \{(a_t)'x \leq b_t, t \in T\}$, (i.e., those linear systems which have the same space of variables, $\mathbb{R}^n$, and the same index set, T). We consider in Y_L the topology of the uniform convergence on T given by the pseudometric

$$\rho_{Y_L}(y_1, y_2) := \sup_{t \in T} \left\| \begin{pmatrix} a_t^1 \\ b_t^1 \end{pmatrix} - \begin{pmatrix} a_t^2 \\ b_t^2 \end{pmatrix} \right\|_\infty,$$

for $y_i = \{(a_t^i)'x \leq b_t^i, t \in T\} \in Y_L$ $i = 1, 2$.

If $y_0 = \left\{f_t^0(x) \leq 0, t \in T\right\}$ is convex, we associate with y_0 the space Y_C of all the convex systems of the form $y = \{f_t(x) \leq 0, t \in T\}$, with $f_t : \mathbb{R}^n \to \mathbb{R}$ convex for all $t \in T$. As in [9], we define a pseudometric ρ_{Y_C} as follows: given $y_i = \left\{f_t^i(x) \leq 0, t \in T\right\} \in Y_C$ $i = 1, 2$,

$$\rho_{Y_C}(y_1, y_2) := \sup_{t \in T} \delta\left(f_t^1, f_t^2\right), \tag{3.1}$$

where

$$\delta\left(f_t^1, f_t^2\right) := \sum_{k=1}^{\infty} 2^{-k} \frac{\delta_k\left(f_t^1, f_t^2\right)}{1 + \delta_k\left(f_t^1, f_t^2\right)},$$

with

$$\delta_k\left(f_t^1, f_t^2\right) := \sup_{\|x\| \leq k} \left|f_t^1(x) - f_t^2(x)\right|, \quad k = 1, 2, \ldots.$$

If a sequence $\{y_r\} \subset Y_C$ satisfies $y_r \to y_0$, with $y_r = \left\{f_t^r(x) \le 0, t \in T\right\} \in Y_C$ $r = 1, 2, \ldots$, then for all $x \in \mathbb{R}^n$, $f_\cdot^r(x) \to f_\cdot^0(x)$ uniformly on T. Moreover $f_t^r \to f_t^0$ uniformly over compact sets in $\mathbb{R}^n$ because of the convexity of the functions f_t^r and f_t^0. It is easy to prove that the topology induced by ρ_{Y_C} on Y_L coincides with the topology associated with the pseudometric of the uniform convergence on T given by d_{Y_L}.

In the general case, $y_0 = \left\{f_t^0(x) \le 0, t \in T\right\}$, the index set T is arbitrary and the functions to be considered, $f_t^0 : \mathbb{R}^n \to \mathbb{R}$, are bounded over bounded sets (not necessarily convex functions). We associate with y_0 the space Y of all the systems of the form $y = \{f_t(x) \le 0, t \in T\}$, where $f_t : \mathbb{R}^n \to \mathbb{R}$ is any function bounded over bounded sets, for each $t \in T$. Finally, we consider in Y the topology induced by the pseudometric ρ_Y defined in the same way as for the space Y_C, see (3.1).

The topology of Y_L is the topology induced Y_C and the topology of Y_C is the one induced by Y. These facts yield that all the sufficient conditions for the stability properties of multivalued mappings defined on Y_C (Y) are inherited by their restrictions to Y_L (Y_C). Nonetheless, the necessary conditions may require a direct argument. We represent by $\mathcal{F}$ the feasible set mapping for Y_L or Y_C or Y in either case.

In a similar way, the space of parameters associated with z_0, Z_L or Z_C or Z, is equipped with the pseudometric d_{Z_L} or ρ_{Z_C} or ρ_Z, respectively.

The following two remarks illustrate conditions **A** and **B** in the case of inequality systems:

Remark 3.1 If $y_0 = \left\{f_t^0(x) \le 0, t \in T\right\}$ is a system such that $\mathcal{F}(y_0)$ is convex and the family of function constraints, $\left\{f_t^0, t \in T\right\}$, is equilipschitzian, then $\mathcal{F}$ satisfies the condition **A**. Indeed, if S is any closed half-space $\omega'(x - u) \le 0$ such that $\|\omega\| = 1$, $u \in \operatorname{bd} \mathcal{F}(y_0)$ and $\mathcal{F}(y_0) \subset S$, the sequence $\{y_r\} \subset Y$,

$$y_r := \left\{f_t^0\left(x - \frac{\omega}{r}\right) \le 0, t \in T\right\}, \ r \in \mathbb{N}, \tag{3.2}$$

verifies that $y_r \to y_0$ and

$$\mathcal{F}(y_r) \setminus S \neq \emptyset \quad \text{for all } r \in \mathbb{N},$$

because $x_r := u + \frac{\omega}{r} \in \mathcal{F}(y_r)$ and

$$\omega'(x_r - u) = \omega'\left(u + \frac{\omega}{r} - u\right) = \frac{1}{r}\|\omega\|^2 = \frac{1}{r} > 0,$$

so $x_r \notin S$.

Observe that for a convex system $\left\{f_t^0(x) \le 0, t \in T\right\}$, the family of the function constraints is equilipschitzian whenever the set of all the corresponding subgradients is bounded because, if $M > 0$ satisfies that $\cup_{t \in T} \partial f_t^0(\mathbb{R}^n) \subset B(0_n; M)$, then the inequality

$$f_t^0(x_1) - f_t^0(x_2) \ge u_t'(x_1 - x_2) \ge -M\|x_1 - x_2\|,$$

for any $x_1, x_2 \in \mathbb{R}^n$ (here u_t is a subgradient of f_t^0 at x_2), yields that

$$\left|f_t^0(x_1) - f_t^0(x_2)\right| \le M\|x_1 - x_2\|.$$

Remark 3.2 $\mathcal{G}$ satisfies condition **B** when $z_0 = \left\{g_t^0(x) \le 0, t \in T\right\}$ $\left\{g_t^0, s \in S\right\}$ is an equilipschitzian family and $\mathcal{G}(z_0)$ is convex. To see this, let H be any supporting hyperplane to $\mathcal{G}(z_0)$. Let $\omega'(x - v) = 0$ be the equation of H, with $\|\omega\| = 1$, $v \in \operatorname{bd} \mathcal{G}(z_0)$ and $\omega'(x - v) \le 0$ for all $x \in \mathcal{G}(z_0)$. Then the sequence $\{z_r\} \subset Z$,

$$z_r := \left\{g_s^0\left(x + \frac{\omega}{r}\right) \le 0, s \in S\right\}, \quad r \in \mathbb{N}, \tag{3.3}$$

satisfies $z_r \to z_0$ and $\mathcal{G}(z_0) = \frac{\omega}{r} + \mathcal{G}(z_r)$. Thus

$$x \in \mathcal{G}(z_r) \Rightarrow x + \frac{\omega}{r} \in \mathcal{G}(z_0),$$

which implies that $\omega' \left(x + \frac{\omega}{r} - v\right) \leq 0$, so that

$$\omega'(x - v) \leq -\frac{1}{r}\|\omega\|^2 < 0.$$

Hence

$$\mathcal{G}(z_r) \cap H = \emptyset \quad \text{for all } r \in \mathbb{N}.$$

Therefore, $\mathcal{G}$ satisfies condition **B**.

Now, we will start with the general case. For non-convex systems it is not possible to relax the assumptions in Theorem 2.2. To get some necessary conditions we could have applied Theorem 2.7, but we prefer for practical uses to replace assumptions **A** and **B** by some conditions easier to verify, getting the stronger necessary condition $d(\mathcal{F}(y_0), \text{bd}\,\mathcal{G}(z_0)) > 0$, which is not true in general as Examples 2.8 and 2.9 show.

Theorem 3.3 *Let* $(y_0, z_0) \in Y \times Z$ *be such that* $\mathcal{G}(z_0)$ *is convex and at least one of the two families of function constraints,* $\{f_t^0, t \in T\}$ *and* $\{g_s^0, s \in S\}$*, is equilipschitzian. If* $\mathcal{F} \subset \mathcal{G}$ *stably at* (y_0, z_0)*, then* $d(\mathcal{F}(y_0), \text{bd}\,\mathcal{G}(z_0)) > 0$ *(and so,* $\mathcal{F}(y_0) \subset \text{int}\,\mathcal{G}(z_0)$*).*

Proof Assume that there exists $M > 0$ such that

$$\left|f_t^0(x_1) - f_t^0(x_2)\right| \leq M\|x_1 - x_2\|,$$

for all $x_1, x_2 \in \mathbb{R}^n$. Take $\delta > 0$ such that $\mathcal{F}(y) \subset \mathcal{G}(z)$ for any $(y, z) \in Y \times Z$ with $\rho_Y(y, y_0) < \delta$ and $\rho_Z(z, z_0) < \delta$. We will use the fact (proved below, at the end) that if $w \in \mathbb{R}^n$ and $y = \{f_t^0(x - w) \leq 0, t \in T\}$ then

$$\rho_Y(y, y_0) \leq \frac{M\|w\|}{1 + M\|w\|}. \tag{3.4}$$

In particular, for any $\omega \in \mathbb{R}^n$, $\|\omega\| = 1$, if $y_r := \left\{f_t^0\left(x - \frac{\omega}{r}\right) \leq 0, t \in T\right\}$, $r \in \mathbb{N}$, we get

$$\rho_Y(y_r, y_0) \leq \frac{M\left\|\frac{\omega}{r}\right\|}{1 + M\left\|\frac{\omega}{r}\right\|} = \frac{M}{r + M}.$$

Take $r_0 \in \mathbb{N}$ such that $\frac{M}{r_0+M} < \delta$ and let $\varepsilon = \frac{1}{2r_0} > \varepsilon$. Suppose that the distance $d(\mathcal{F}(y_0), \text{bd}\,\mathcal{G}(z_0)) = 0$. Consider $u \in \mathcal{F}(y_0)$ and $v \in \text{bd}\,\mathcal{G}(z_0)$ such that $\|u - v\| < \varepsilon$. By the supporting hyperplane theorem we can choose some $\omega \in \mathbb{R}^n$, $\|\omega\| = 1$, such that

$$\omega'(x - v) \leq 0 \quad \text{for all } x \in \mathcal{G}(z_0). \tag{3.5}$$

For y_{r_0} we have that $\rho_Y\left(y_{r_0}, y_0\right) < \delta$, and so $\mathcal{F}\left(y_{r_0}\right) \subset \mathcal{G}(z_0)$ by the choice of δ. Since $\mathcal{F}\left(y_{r_0}\right) = \mathcal{F}(y_0) + \frac{\omega}{r_0}$ and $u \in \mathcal{F}(y_0)$, it follows that $u + \frac{\omega}{r_0} \in \mathcal{G}(z_0)$. Then, by (3.5),

$$0 \geq \omega'\left(u + \frac{\omega}{r_0} - v\right) = \frac{1}{r_0}\|\omega\|^2 + \omega'(u - v) \geq \frac{1}{r_0} - \|u - v\| > \frac{1}{r_0} - \varepsilon = \frac{1}{2r_0} > 0,$$

a contradiction. Therefore it must be that $d(\mathcal{F}(y_0), \text{bd}\,\mathcal{G}(z_0)) > 0$.

In case the equilipschitzian family is the one corresponding to the mapping $\mathcal{G}$, the result follows in the same fashion as above by considering the sequence given by $z_r := \left\{g_s^0\left(x + \frac{\omega}{r}\right) \leq 0, s \in S\right\}$.

To finish the proof we only need to show the assertion (3.4). For any $k \in \mathbb{N}$,

$$\delta_k\left(f_t^0(\cdot), f_t^0(\cdot - w)\right) := \max_{\|x\| \leq k} \left|f_t^0(x) - f_t^0(x - w)\right| \leq M\|w\|.$$

Since the function $s \mapsto \frac{s}{1+s}$ is increasing on $\mathbb{R}_+$, for any $t \in T$, we have

$$\begin{aligned}\delta\left(f_t^0(\cdot), f_t^0(\cdot - w)\right) &:= \sum_{k=1}^{\infty} 2^{-k} \frac{\delta_k\left(f_t^0(\cdot), f_t^0(\cdot - w)\right)}{1 + \delta_k\left(f_t^0(\cdot), f_t^0(\cdot - w)\right)} \\ &\leq \sum_{k=1}^{\infty} 2^{-k} \frac{M\|w\|}{1 + M\|w\|} \\ &= \frac{M\|w\|}{1 + M\|w\|},\end{aligned}$$

so that

$$\rho_Y(y, y_0) := \sup_{t \in T} \delta\left(f_t^0(\cdot), f_t^0(\cdot - w)\right) \leq \frac{M\|w\|}{1 + M\|w\|}.$$

This completes the proof. □

Theorem 3.4 *Let y_0 and z_0 be convex systems such that $\mathcal{F}(y_0) \neq \emptyset$. Then the following statements hold:*

(i) *If $\mathcal{F}(y_0) \subset \operatorname{int} \mathcal{G}(z_0)$, $\mathcal{G}$ is lsc at z_0, and $\mathcal{G}, (z_0)$ is bounded, then $\mathcal{F} \subset \mathcal{G}$ stably at (y_0, z_0).*

(ii) *If $\mathcal{F} \subset \mathcal{G}$ stably at (y_0, z_0) and the set of subgradients of the function constraints is bounded for at least one of them, then $d(\mathcal{F}(y_0), \operatorname{bd} \mathcal{G}(z_0)) > 0$ and $\mathcal{G}$ is lsc at z_0.*

Proof

(i) Since $\mathcal{F}(y_0)$ and $\mathcal{G}(z_0)$ are bounded, $\mathcal{F}$ and $\mathcal{G}$ are usc at y_0 and z_0, respectively. The conclusion follows from Corollary 2.3.

(ii) The argument to show that $d(\mathcal{F}(y_0), \operatorname{bd} \mathcal{G}(z_0)) > 0$ is the same as in the previous theorem. To prove that $\mathcal{G}$ is lsc at z_0, we use the fact that for convex systems the lower semicontinuity property at a consistent system is equivalent to the stability with respect to the consistency [9, Theorem 4.1]. Now, z_0 is in the interior of the domain of $\mathcal{G}$ because, by the assumptions, $\emptyset \neq \mathcal{F}(y_0) \subset \mathcal{G}(z)$ for any z closed enough to z_0, hence $\mathcal{G}$ is lsc at z_0. □

Different characterizations of the lsc property of $\mathcal{G}$ at z_0 that can be checked in terms of the data have been given in [9]. For ordinary (finite) systems the boundedness condition in (ii) can be replaced by the lipschitzian property of the function constraints of any one of the involved systems.

The event of the family of the function constraints of the linear system $\left\{a_t' x \leq b_t, \ t \in T\right\}$ is even easier because

$$\left\|\left(a_t' x_1 - b_t\right) - \left(a_t' x_2 - b_t\right)\right\| \leq \|a_t\| \|x_1 - x_2\| \leq M\|x_1 - x_2\|,$$

if $\|a_t\| \leq M$ for all $t \in T$. Hence, together with Corollary 2.3, we get the following result for linear systems ([5, Proposition 4.1]):

Theorem 3.5 *Let y_0 and z_0 be linear systems such that $\mathcal{F}(y_0) \neq \emptyset$. Then the following statements hold*:

(i) *If $\mathcal{F}(y_0) \subset$ int $\mathcal{G}(z_0)$, $\mathcal{G}$ is lsc at z_0 and $\mathcal{G}(z_0)$ is bounded, then $\mathcal{F} \subset \mathcal{G}$ stably at (y_0, z_0).*
(ii) *If $\mathcal{F} \subset \mathcal{G}$ stably at (y_0, z_0) and the set of gradients of either y_0 or z_0 is bounded, then $\mathcal{F}(y_0) \subset$ int $\mathcal{G}(z_0)$ and $\mathcal{G}$ is lsc at z_0.*

In LSISs, the boundedness of $\mathcal{G}(z_0)$ in (i) can be replaced by the weaker characterization of the usc property of $\mathcal{G}$ that can be found in [3], with the inconvenient that this property is not inherited by $\mathcal{F}$ and, moreover, it can hardly be checked in practice. In the particular case that z_0 is an ordinary linear system, $\mathcal{G}$ is lsc at z_0 if and only if the Slater condition holds and it is usc at z_0 if and only if $\mathcal{G}(z_0)$ is either bounded or the whole space $\mathbb{R}^n$.

Corollary 3.6 *Let y_0 and z_0 be ordinary linear systems such that $\mathcal{F}(y_0) \neq \emptyset$. Then the following statements hold*:

(i) *If $\mathcal{F}(y_0) \subset$ int $\mathcal{G}(z_0)$ and $\mathcal{G}$ is continuous at z_0, then $\mathcal{F} \subset \mathcal{G}$ stably at (y_0, z_0).*
(ii) *If $\mathcal{F} \subset \mathcal{G}$ stably at (y_0, z_0), then $\mathcal{F}(y_0) \subset$ int $\mathcal{G}(z_0)$ and $\mathcal{G}$ is lsc at z_0.*

We have assumed in this section that y_0 and z_0 are systems of the same class. However, analogous results for mixed combinations can be obtained in a similar way.

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